MAGNETOHYDRODYNAMIC FLOWS IN PIPES IN PRESENCE OF A LONGITUDINAL CURRENT

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PMM Vol. 31, No. 2, 1967, pp. 356-362

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(Received December 12, 1966)

This paper supplements the results obtained in [1]. We consider in it some general properties of fully developed magnetohydrodynamic flows in pipes in presence of a longitudinal electric current. Examples of exact solutions given here show, that the thickness of a boundary layer on the walls parallel to the direction of field, are of the order of $M^{-\frac{1}{2}}$ where M is a Hartmann number.

1. Fully developed steady flow of an incompressible, isotropically conducting fluid in a pipe, was investigated previously in [1 to 3]. It was shown that transverse distributions of velocity $u(y, z)$ and induced magnetic field $b(y, z)$ satisfy Eqs. (1.1)

$$\eta \triangle u + \frac{1}{4\pi} \mathbf{B}_\perp \nabla b = -P^*, \qquad \nu_m \triangle b + \mathbf{B}_\perp \nabla u = 0, \qquad P^* \equiv -\frac{\partial p^*}{\partial x} = \text{const}$$

with the corresponding boundary conditions, provided that the following equalities,

$$\operatorname{div} \mathbf{B}_\perp = 0, \qquad \operatorname{rot} \mathbf{B}_\perp = \frac{4\pi}{c} j_x \mathbf{e}_x, \qquad j_x = \text{const} \tag{1.2}$$

where j_x is a specified constant, hold for a magnetic field $\mathbf{B}_\perp(y, z)$ transverse to the flow. Hydrodynamic pressure $p = p^* - B^2 / 8\pi$ is given by

$$p + \frac{b^2}{8\pi} = Cx - \frac{1}{c} A j_x \qquad (C = \text{const}) \tag{1.3}$$

where A is vector potential of the field $\mathbf{B}_\perp$, i.e. $B_z = \partial A / \partial y$, $B_y = -\partial A / \partial z$.

The conditions imposed on $\mathbf{B}_\perp$ by (1.2) are not sufficient for the solution of (1.1) and (1.3) to acquire a physical meaning. Indeed, if A is a multivalued function of y and z, then the pressure p will not be singlevalued when $j_x \neq 0$, although the field $\mathbf{B}_\perp$ may be uniquely defined. Lack of uniqueness of the pressure implies that in a pipe of given profile another motion in the (y, z)-plane takes place, apart from the main flow. For example, in an annular pipe with a radial magnetic field and axial current, the resulting motion is not purely longitudinal, but always spiral (see e.g. [4]).

Unique determination of pressure is possible if the integral of $\partial p / \partial s$ over any closed contour L lying entirely within the cross section of the pipe, is equal to zero. From this, taking into account (1.3) and the fact that $\partial A / \partial s$ is proportional to the component of $\mathbf{B}_\perp$ normal to the contour, we obtain

$$\oint_L B_{n\perp}\, ds = 0 \tag{1.4}$$

Physical significance of the condition (1.4) is obvious. Total pole strength inside the fluid and in the internal cavities of the pipe, should be equal to zero. Our previous example of a flow in an annular pipe contradicts this requirement, since the source of radial field is situated within an internal cavity. However, if the radial field is replaced with a homogeneous transverse field, then the condition (1.4) will be fulfilled and we shall be able to construct a solution to (1.1) and (1.3). Thus we can see that Eqs. (1.2) and (1.4) form a system of conditions necessary and sufficient for the existence of a fully developed flow.

It can easily be shown that (1.1) has a simple particular solution

$$u = \text{const} - \frac{cP^*A}{4\pi\eta j_x} \qquad b = \text{const} \tag{1.5}$$

If the external magnetic field is such that the above expressions for velocity and the field satisfy all the boundary conditions, then the flow differs from the usual hydrodynamic flow only in its pressure distribution and in the appearance of an electric field. A flow in a nonconducting pipe of circular cross section with longitudinal current and in absence of other sources of magnetic field [5] is an example of such a trivial situation. Below we consider some nontrivial problems in which velocity distribution varies appreciably.

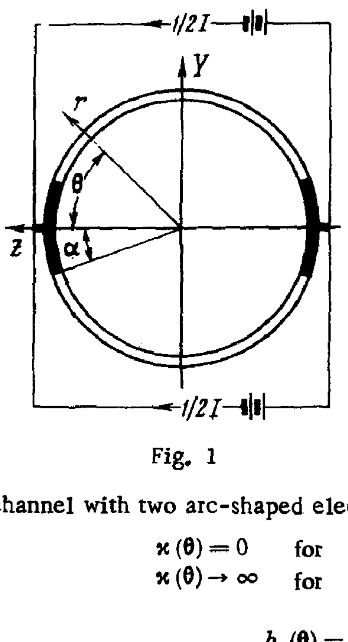

Fig. 1

2. Let the current j_x be the only source of transverse field during a flow in a pipe of circular cross section of radius r_0, so that

$$\mathbf{B}_\perp = \mathbf{e}_\theta B_\theta, \quad B_\theta = 2\pi j_x r/c, \quad A = \pi j_x r^2/c$$

We shall assume the walls of a pipe to be thin and ideally segmented, i.e. possessing zero conductivity in the x-direction and arbitrary conductivity in the θ-direction. Then [5]

$$u = 0, \qquad b - b_e(\theta) = -\varkappa(\theta)\ \partial b/\partial r \quad \text{for } r = r_0 \tag{2.1}$$

Here $b_e(\theta)$ is the longitudinal field outside the pipe and $\varkappa(\theta)$ is the corresponding azimuthal conductivity of the wall. For example, in case of a channel with two arc-shaped electrodes (Fig. 1), we have

$$\begin{gathered}
\varkappa(\theta) = 0 \quad \text{for} \quad \alpha < \theta < \pi - \alpha, \quad \pi + \alpha < \theta < 2\pi - \alpha \\
\varkappa(\theta) \to \infty \quad \text{for} \quad -\alpha < \theta < \alpha, \quad \pi - \alpha < \theta < \pi + \alpha \\
b_e(\theta) = \begin{cases} 2\pi I/c & \text{for } 0 < \theta < \pi \\ -2\pi I/c & \text{for } \pi < \theta < 2\pi \end{cases}
\end{gathered} \tag{2.2}$$

When $b_e \equiv 0$ and $\varkappa(\theta)$ is finite, solution (1.5) satisfies the conditions (2.1), provided that the constants are suitably selected.

When $b_e \not\equiv 0$, we shall for simplicity assume that

$$b(r_0, \theta) = b_w(\theta) \tag{2.3}$$

is a given function and $b_w(0) = b_w(\pi)$ while $b_w(-\theta) = -b_w(\theta)$.

Let us now write (1.1), (2.1) and (2.3) in dimensionless variables, choosing

$$r_0, \qquad U = \frac{1}{\pi r_0^2}\int_0^{r_0}\int_{-\pi}^{\pi} r u\, d\theta\, dr, \qquad \frac{4\pi\sqrt{\sigma\eta}}{c^2}U$$

as characteristic units of length, velocity and the field, and retaining the previous designation for u and b. Then,

$$\Delta u + M\partial b/\partial\theta = -P, \qquad \Delta b + M\partial u/\partial\theta = 0 \tag{2.4}$$

$$u(1, \theta) = 0, \qquad b(1, \theta) = b_w(\theta) \tag{2.5}$$

$$P = P^* \frac{r_0^2}{U\eta}, \quad M = \frac{2\pi j_x r_0^2}{c^2}\left(\frac{\sigma}{\eta}\right)^{1/2}, \quad \Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$$

Solution of (2, 4) and (2, 5) will be sought in the form

$$u = \frac{P}{4}(1 - r^2) + \sum_{k=1}^{\infty} u_k(r)\cos k\theta, \qquad b = \sum_{k=1}^{\infty} b_k(r)\sin k\theta \tag{2.6}$$

$$(D_k^2 + k^2M^2)\, u_k = 0, \qquad (D_k^2 + k^2M^2)\, b_k = 0 \tag{2.7}$$

$$u_k(1) = 0, \qquad b_k(1) = \beta_k = \frac{1}{\pi}\int_{-\pi}^{\pi} b_w \sin k\theta\, d\theta \tag{2.8}$$

$$D_k u_k(1) = -kM\beta_k, \qquad D_k b_k(1) = 0, \qquad D_\nu = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{\nu^2}{r^2}$$

Solutions (2, 7) and (2, 8) of boundary value problems regular within the circle $0 \leq r \leq 1$, can be expressed in terms of Thomson functions

$$u_k = -\beta_k \frac{\mathrm{bei}_k r\sqrt{kM}\,\mathrm{ber}_k\sqrt{kM} - \mathrm{bei}_k\sqrt{kM}\,\mathrm{ber}_k r\sqrt{kM}}{\mathrm{ber}_k^2\sqrt{kM} + \mathrm{bei}_k^2\sqrt{kM}}$$

$$b_k = \beta_k \frac{\mathrm{ber}_k r\sqrt{kM}\,\mathrm{ber}_k\sqrt{kM} + \mathrm{bei}_k r\sqrt{kM}\,\mathrm{bei}_k\sqrt{kM}}{\mathrm{ber}_k^2\sqrt{kM} + \mathrm{bei}_k^2\sqrt{kM}} \tag{2.9}$$

Velocity distribution (2, 6) and (2, 9) exhibits some properties associated with flows under the conditions of free convection [6], e. g. mean velocity $U = P^* r_0^2 / 8\eta$ and the axial velocity $u(0) = 2U$, are independent of the Hartmann number M.

When the function $b_w(\theta)$ with its Fourier coefficients β_k is given (this is equivalent to specifying a radial current $j_r \sim b_w'$ on the boundary), then Formulas (2, 6) and (2, 9) give complete solution to the problem and (2, 1) yields the connection between the corresponding distributions of $b_e(\theta)$ and $\varkappa(\theta)$. If, on the other hand, b_w is defined in terms of $b_e(\theta)$ and $\varkappa(\theta)$, then (2, 1) yields, for the coefficients β_k, an infinite system of linear equations which can be solved by numerical methods (when $\varkappa(\theta)$ is given in the form of (2, 2), we have a so called problem on dual trigonometric series)

$$\beta_l + \frac{1}{2}\sum_{k=1}^{\infty} \beta_k \gamma_k (\varkappa_{k-l} - \varkappa_{k+l}) = \varepsilon_l \qquad (l = 1, 2, \ldots)$$

$$\varepsilon_l = \frac{1}{\pi}\int_{-\pi}^{\pi} b_e \sin l\theta\, d\theta, \qquad \varkappa_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \varkappa \cos k\theta\, d\theta \tag{2.10}$$

$$\gamma_k \equiv \beta_k^{-1} b_k'(1) = -\left(\sqrt{1/2 kM} / \Delta_k\right)\left[\sqrt{2k/M}\,\Delta_k + \mathrm{ber}_k\sqrt{kM}\,\mathrm{ber}_{k-1}\sqrt{kM} + \right.$$

$$\left. + \mathrm{bei}_k\sqrt{kM}\,\mathrm{bei}_{k-1}\sqrt{kM} - \mathrm{bei}_k\sqrt{kM}\,\mathrm{ber}_{k-1}\sqrt{kM} + \mathrm{ber}_k\sqrt{kM}\,\mathrm{bei}_{k-1}\sqrt{kM}\right] \tag{2.11}$$

$$\Delta_k = \mathrm{ber}_k^2\sqrt{kM} + \mathrm{bei}_k^2\sqrt{kM}$$

We shall use asymptotic representations of Thomson functions [7]

$$\mathrm{ber}_k z \sim \frac{e^{z/\sqrt{2}}}{\sqrt{2\pi z}}\cos\left(\frac{z}{\sqrt{2}} + \frac{k\pi}{2} - \frac{\pi}{8}\right), \qquad \mathrm{bei}_k z \sim \frac{e^{z/\sqrt{2}}}{\sqrt{2\pi z}}\sin\left(\frac{z}{\sqrt{2}} + \frac{k\pi}{2} - \frac{\pi}{8}\right)$$

to study the obtained solution when $r\sqrt{M} \gg 1$, and we shall transform (2.9) and (2.11) into

$$u_k \sim \beta_k \frac{1}{\sqrt{r}} \exp[(r-1)\sqrt{kM/2}] \sin(r-1)\sqrt{kM/2},$$

$$b_k \sim \beta_k \frac{1}{\sqrt{r}} \exp[(r-1)\sqrt{kM/2}] \cos(r-1)\sqrt{kM/2}, \qquad \gamma_k \sim \sqrt{kM/2} \tag{2.12}$$

Inspection of these formulas shows that near the wall $r = 1$ boundary layers (dynamic and electric) are formed and that their thickness decreases with increasing Hartmann number as $M^{-\frac{1}{2}}$, provided that β_k are fixed. We also see that when $M \to \infty$ and when $M \to 0$, the flow tends to become an ordinary Poiseuille flow. Therefore, there exists some value of M, corresponding to the greatest symmetry in the flow and the greatest value of

$$W = \sum_{k=1}^{\infty} \int_0^1 r u_k^2 \, dr$$

can be taken as the measure of this asymmetry.

Formulas (2.12) show also that periodic variation in velocity and field is possible near the wall of the pipe.

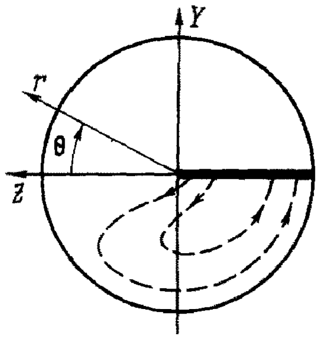

Fig. 2

Having calculated the components j_θ and j_r of the current density in terms of derivatives of b and having investigated their asymptotic behavior at large M, we can easily establish that the radial component j_r decreases exponentially with increasing M, when $r < 1$. Azimuthal component j_θ also decreases in this case, but $\sqrt{M}$ times slower, and at $r = 1$, increases infinitely as $\sqrt{M}$. Thus, when $M \gg 1$ the flow narrows into a boundary layer and becomes parallel to the external field. This explains its tendency to become a Poiseuille flow as $M \to \infty$.

3. In the previous problem, nontriviality of the solution, i.e. deviation of the velocity profile from Poiseuillian was caused by the inhomogeneity of longitudinal magnetic field on the boundary. Obviously, the flow will not be Poiseuillian under any violation of symmetry of the boundary conditions. Let us for example assume, that a cylindrical pipe has a radial partition (Fig. 2) whose longitudinal conductivity is equal to that of the fluid, while the radial conductivity is infinite. Walls of the pipe will, for simplicity, be assumed nonconducting. If the magnetic field is, as in Section 2, generated only by the constant current j_x, then the distributions of velocity and induced field in a flow taking place in such a channel, are found from the solution of Eqs. (2.4) with the following boundary conditions

$$u = b = 0 \quad \text{for } r = 1, \; \partial u/\partial\theta = b = 0 \quad \text{for } \theta = 0$$
$$\partial b/\partial\theta = u = 0 \quad \text{for } \theta = \pm\pi \tag{3.1}$$

taken into account.

Putting

$$u = \sum_{k=1}^{\infty} u_k(r) \cos\left(k - \frac{1}{2}\right)\theta, \qquad b = \sum_{k=1}^{\infty} b_k(r) \sin\left(k - \frac{1}{2}\right)\theta \tag{3.2}$$

we obtain

$$[D_{k-1/2}^2 + (k - 1/2)^2 M^2]\, u_k = -\gamma_k (k - 1/2)^2 r^{-2}, \qquad u_k(1) = 0, \qquad D_{k-1/2} u_k(1) = -\gamma_k$$
$$\gamma_k = \frac{P}{\pi}\int_{-\pi}^{\pi} \cos (k - 1/2)\,\theta\, d\theta = \frac{2(-1)^{k+1}}{\pi (k - 1/2)}, \qquad b_k = -\frac{\gamma_k + D_{k-1/2} u_k}{M (k - 1/2)} \tag{3.3}$$

Let us now introduce the operators

$$L_k = D_{k-1/2} + (k - 1/2)\, M e^{-3/2\pi i}, \quad \bar{L}_k = D_{k-1/2} + (k - 1/2)\, M e^{3/2\pi i} \tag{3.4}$$

The relations

$$L_k + \bar{L}_k = 2 D_{k-1/2}, \qquad L_k - \bar{L}_k = \bar{\tau}_k^2 - \tau_k^2, \qquad L_k \bar{L}_k = \bar{L}_k L_k \tag{3.5}$$
$$\tau_k = e^{3/4\pi i}\sqrt{(k - 1/2) M}$$

are obvious.

Writing (3.3) in the form $L_k \bar{L}_k u_k = -\gamma_k (k - 1/2)^2 r^{-2}$ and remembering that the solution is bounded when $r = 0$, we obtain

$$\bar{L}_k u_k = F_k J_{k-1/2}(\bar{\tau}_k r) + \bar{\tau}_k^2 \gamma_k (k - 1/2)^2 s_{-3,\, k-1/2}(\bar{\tau}_k r)$$
$$L_k u_k = E_k J_{k-1/2}(\tau_k r) + \tau_k^2 \gamma_k (k - 1/2)^2 s_{-3,\, k-1/2}(\tau_k r)$$

where E_k and F_k are constants, while $s_{-3,k-1/2}$ are Lommel functions. This, together with the second Eq. of (3.5) yields u_k which, with (3.4) and the first Eq. of (3.5), gives b_k. Adding the conditions (3.3), we obtain the final form

$$u_k = \frac{\gamma_k}{M (k - 1/2)} \operatorname{Im}\left[\frac{J_{k-1/2}(\bar{\tau}_k r)}{J_{k-1/2}(\bar{\tau}_k)}\right] -$$
$$- \gamma_k (k - 1/2)^2 \operatorname{Re}\left[s_{-3,\, k-1/2}(\tau_k)\, \frac{J_{k-1/2}(\bar{\tau}_k r)}{J_{k-1/2}(\bar{\tau}_k)} - s_{-3,\, k-1/2}(\tau_k r)\right]$$
$$b_k = \frac{\gamma_k}{M (k - 1/2)} \left[1 - \operatorname{Re}\frac{J_{k-1/2}(\bar{\tau}_k r)}{J_{k-1/2}(\bar{\tau}_k)}\right] +$$
$$+ \gamma_k (k - 1/2)^2 \operatorname{Im}\left[s_{-3,\, k-1/2}(\tau_k)\, \frac{J_{k-1/2}(\bar{\tau}_k r)}{J_{k-1/2}(\bar{\tau}_k)} - s_{-3,\, k-1/2}(\tau_k r)\right]$$

Transition to Thomson functions and their integrals is easy, but uninteresting. Use of asymptotic representations of Bessel and Lommel functions [8] for $r\sqrt{M} \gg 1$ enables us, as in the previous problem, to establish the formation of a boundary layer of thickness of the order of $M^{-1/2}$ near the wall, and to confirm the fact that the distribution of u and b in r, can be nonmonotonous.

A Hartmann type boundary layer is formed on the partition $\theta = \pm\pi$, $r > 0$. Its thickness can be found from (2.4) and (3.1) by introducing a new variable $\xi = r\theta$, neglecting the derivatives in r as compared with those in ξ and obtaining a solution in its general form. As expected, thickness of the boundary layer is found to be of the order of $(rM)^{-1}$.

4. When the axial symmetry of transverse magnetic field $B_\perp$ is disturbed, the resulting flows are non-Poiseuillian.

Consider e. g. the case when a homogeneous external field (here, as opposed to previous problems, polar angle θ is taken from the y-axis)

$$\mathbf{B}_\perp = \mathbf{e}_\theta (2\pi j_x r/c) - \mathbf{e}_z B_0 = \mathbf{e}_r B_0 \sin\theta + \mathbf{e}_\theta (B_0 \cos\theta + 2\pi j_x r/c) \tag{4.1}$$

is superimposed on the azimuthal field of the longitudinal current. Such a field satisfies the condition (1, 4). Corresponding developed flow in a circular pipe with nonconducting walls, is described by (4.2)

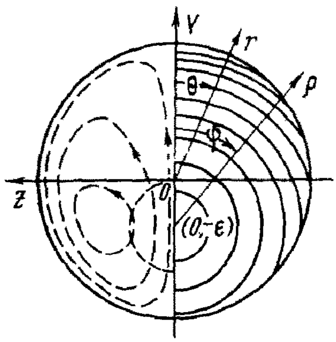

Fig. 3

$$\Delta u + M\left[\varepsilon \frac{\partial b}{\partial r}\sin\theta + \frac{\partial b}{\partial \theta}\left(\frac{\varepsilon}{r}\cos\theta + 1\right)\right] = -P$$

$$\Delta b + M\left[\varepsilon \frac{\partial u}{\partial r}\sin\theta + \frac{\partial u}{\partial \theta}\left(\frac{\varepsilon}{r}\cos\theta + 1\right)\right] = 0$$

$$u(1, \theta) = b(1, \theta) \qquad (4.3)$$

where $\varepsilon = B_{0c} / 2\pi j_x r_0$, and the remaining dimensionless magnitudes are the same as in Section 2.

Lines of force of the transverse magnetic field $B_\perp$ (solid lines on Fig. 3) form a family of concentric circles with the center at the point $\theta = \pi$, $r = \varepsilon$. As ε increases from zero, the center of lines of force moves away from the center of cross section of the pipe, in the negative direction of the y-axis. Thus, when $\varepsilon \to 0$, the flow becomes Poiseuillian (see Section 1), while when $\varepsilon \to \infty$, it tends to the well known flow in a homogeneous external field [2 and 5].

Let us now make use of polar coordinates with their center at the point (ε, π). We put

$$\rho^2 = r^2 + 2\varepsilon\, r\cos\theta + \varepsilon^2, \qquad \rho\sin\psi = r\sin\theta, \qquad \rho\cos\psi = r\cos\theta + \varepsilon \qquad (4.4)$$

Using the variables ρ and ψ we obtain, from (4, 2) and (4, 3), a system analogous to (2, 4) and (2, 5)

$$\Delta u + M\frac{\partial b}{\partial \psi} = -P, \qquad \Delta b + M\frac{\partial u}{\partial \psi} = 0 \qquad (4.5)$$

$$u = b = 0 \quad \text{for } \rho^2 = 1 - \varepsilon^2 + 2\varepsilon\rho\cos\psi \qquad (4.6)$$

Taking into account the symmetry in ψ, we can write the solution of (4, 5) as

$$u = \frac{P}{4}(1 - \rho^2) + \sum_{k=1}^{\infty} u_k(\rho)\cos k\psi, \qquad b = \sum_{k=1}^{\infty} b_k(\rho)\sin k\psi \qquad (4.7)$$

where u_k and b_k are given by

$$D_k u_k + kMb_k = 0, \qquad D_k b_k - kMu_k = 0 \qquad (4.8)$$

with accuracy of up to four constants for each k. In this manner we arrive, fulfilling the boundary conditions (4, 6) for infinite sums (4, 7), at an infinite algebraic system. We should note that when $\varepsilon < 1$, then only two out of each four constants appear in the system. The other two are made equal to zero because of the requirement of regularity of the system as $\rho \to 0$. When $\varepsilon > 1$, then all the constants have to be determined and the number of equations is doubled, since ψ varies within finite limits ($|\psi| < \sin^{-1}(1/\varepsilon)$) and two values of ρ correspond to each value of ψ.

Let us limit ourselves to the case $\varepsilon < 1$ and derive for it the algebraic system mentioned above. We shall utilize the solution of Eqs. (4, 8), which has the form

$$u_k(\rho) = E_k J_k(\tau_k\rho) + F_k J_k(\bar{\tau}_k\rho), \quad b_k(\rho) = -iE_k J_k(\tau_k\rho) + iF_k J_k(\bar{\tau}_k\rho) \qquad (4.9)$$

$$\tau_k = e^{3/4\pi i}\sqrt{kM}$$

together with relations

$$\cos k\psi = \frac{1}{\rho^k}\sum_{s=0}^{s=k}\frac{\varepsilon^s r^{k-s}k!}{s!\,(k-s)!}\cos(k-s)\,\theta, \qquad \sin k\psi = \frac{1}{\rho^k}\sum_{s=0}^{s=k}\frac{\varepsilon^s r^{k-s}k!}{s!\,(k-s)!}\sin(k-s)\,\theta \tag{4.10}$$

and the Gegenbauer formula

$$\frac{J_k(\tau_k\rho)}{(\tau_k\rho)^k} = 2^k\sum_{l=0}^{\infty}(k+l)\frac{J_{k+l}(\tau_k r)\,J_{k+l}(-\varepsilon\tau_k)}{(\tau_k r)^k(-\varepsilon\tau_k)^k}\,\Omega_l^{\,k}(\cos\theta) \tag{4.11}$$

$$\Omega_l^{\,k}(p) = \sum_{j=0}^{j\leqslant 1/2 l}\frac{(-1)^j 2^{l-2j}(k+l-j-1)!\;p^{l-2j}}{(l-2j)!\;j!}$$

Inserting (4. 9) to (4. 11) into series (4. 7), putting $r = 1$, $u = 0$ and $b = 0$ and multiplying respectively by $\cos m\theta$ and $\sin m\theta$ we obtain, on integration from $-\pi$ to π, an infinite system in E_k and F_k

$$\sum_{k=1}^{\infty}\sum_{l=0}^{\infty}(A_{kl}E_k + \overline{A}_{kl}F_k)\sum_{s=0}^{k}\sum_{j=0}^{\leqslant 1/2 l}P(k,l,s,j)\,[J(\,|k-s-m|,\,l-2j) + J(k-s+m,\,l-2j)] = a_m P \tag{4.12}$$

$$\sum_{k=1}^{\infty}\sum_{l=0}^{\infty}(A_{kl}E_k - \overline{A}_{kl}F_k)\sum_{s=0}^{k}\sum_{j=0}^{\leqslant 1/2 l}P(k,l,s,j)\,[J(\,|k-s-m|,\,l-2j) - J(k-s+m,\,l-2j)] = 0$$

$$P(k,l,s,j) = \frac{(-1)^j 2^{l-2j+k}\varepsilon^s(k+l-j-1)!\;k!\;(k+l)}{(l-2j)!\;j!\;s!\;(k-s)!}, \qquad A_{kl} = \frac{J_{k+l}(\tau_k)\,J_{k+l}(-\varepsilon\tau_k)}{(-\varepsilon\tau_k)^k}$$

$$J(p,q) = \frac{\pi}{2^q}\;\frac{q(q-1)\cdots(q-\lambda+1)}{\lambda!} \qquad \text{for } q-p = 2\lambda > 0 \qquad (p,\,q \text{ are whole numbers})$$

$$J(p,q) = 0 \qquad \text{for } p > q \text{ and for } q-p = 2\lambda-1 > 0$$

$$(m = 0, 1, 2, 3\ldots,\; a_0 = 1/2\,\varepsilon^2\pi,\; a_1 = 1/2\,\varepsilon\pi,\; a_2 = a_3 = \cdots = 0)$$

Obviously, when $\epsilon \to 0$, the right-hand sides vanish and we have a trivial solution $E_k = F_k = 0$ which has, according to (4. 7), a corresponding Poiseuille flow.

System (4. 12) becomes somewhat simpler when $\epsilon \to 1-0$, when the center of the lines of force approaches the wall of the pipe. It can be written as

$$\sum_{k=1}^{\infty}(E_k a_{km}^{+} + F_k\overline{a}_{km}^{+}) = a_m P, \qquad \sum_{k=1}^{\infty}(E_k a_{km}^{-} - F_k\overline{a}_{km}^{-}) = 0$$

$$a_{km}^{\pm} = \pi J_{2m}(\tau_k)[J_{2(k-m)}(\tau_k) \pm J_{2(k+m)}(\tau_k)]$$

$$m = 0, 1, 2, \ldots, \quad a_0 = a_1 = 1/2\,\pi, \quad a_2 = a_3 = \cdots = 0$$

When values of ϵ are small and $\epsilon\sqrt{M} \ll 1$, we can construct a solution in form of a series in powers of ϵ. Eq. (4. 9) and the system (4. 12) then yield, with accuracy of up to the order of ϵ^2,

$$u \approx \frac{P}{4}(1 - r^2 - 2r\varepsilon\cos\theta) + \frac{\varepsilon P}{2}\,\frac{\operatorname{ber}_1 r\sqrt{M}\,\operatorname{ber}_1\sqrt{M} + \operatorname{bei}_1 r\sqrt{M}\,\operatorname{bei}_1\sqrt{M}}{\operatorname{ber}_1^2\sqrt{M} + \operatorname{bei}_1^2\sqrt{M}}\cos\theta \tag{4.13}$$

$$b \approx \frac{\varepsilon P}{2}\,\frac{\operatorname{ber}_1\sqrt{M}\,\operatorname{bei}_1 r\sqrt{M} - \operatorname{ber}_1 r\sqrt{M}\,\operatorname{bei}_1\sqrt{M}}{\operatorname{ber}_1^2\sqrt{M} + \operatorname{bei}_1^2\sqrt{M}}\sin\theta$$

In this approximation, distribution of induced field is symmetric with respect to $y = 0$;

in higher approximations this symmetry disappears.

A complex pattern of currents in the cross section of the pipe is shown schematically on Fig. 3 with broken lines. Direction of circulation is given by the highest e. m. f. (in the present case the e. m. f. when $y > -\epsilon$) and the current density by the difference of absolute values of e. m. f. at $y > -\epsilon$ and $y < -\epsilon$. As $\epsilon \to 0$, that difference also tends to zero and the current disappears. Although the magnetic field is not homogeneous when $\epsilon < 1$, current loops do not show any singularities when compared with the case of a homogeneous field [5]. At large Hartmann numbers a boundary layer is formed near the wall, in which the current loops are closed, while at the center the direction is radial. Near the point (ϵ, π) a region $(r \sim M^{-\frac{1}{2}})$ appears, where the e. m. f. is small and current flows in the y-direction. For small Hartmann numbers, this region becomes a circumference of radius $r \approx \frac{1}{2}\sqrt{3}$. These facts are easily established by considering the asymptotic behavior of the solution (4. 3) at the small and large values of M.

BIBLIOGRAPHY

1. Regirer, S. A., O techenii electroprovodnoi zhidkosti v prisutstvii magnitnogo polia po trubam proizvol'nogo profilia (On the flow of electrically conducting fluid in pipes of arbitrary cross section in the presence of magnetic field). PMM Vol. 24, No. 3, 1960.
2. Iakubenko, A. E., Statsionarnoe techenie viazkoi neszhimaemoi provodiashchei zhidkosti po trubam v odnorodnom i neodnorodnom magnitnom pole (Steady flow of a viscous incompressible conducting fluid in pipes, in a homogeneous and inhomogeneous magnetic field). Izv. Akad. Nauk SSSR, Mekhanika i mashinostroenie. No. 1, 1961.
3. Regirer, S. A., Techenie viazkoi provodiashchei zhidkosti v trubakh pri nalichii magnitnogo polia (Flow of a viscous conducting fluid in pipes in presence of magnetic field). Collection of problems of magnetohydrodynamics and plasma dynamics. Vol. 2, Izd. Akad. Nauk LatvSSR, 1962.
4. Diki, G. P. and Tarapov, I. E., Nekotorye statsionarnye zadachi magnitnoi gidrodinamiki s aksial'noi simmetriei (Some axially symmetric steady state problems of magnetohydrodynamics). Zh. tekh. Fiz., Vol. 32, No. 11, 1962.
5. Shercliff, J. A., Theory of Electromagnetic Flow-Measurement. Cambr. Univ. Press, 1962. (Russian translation, Izd. "Mir", 1965.)
6. Regirer, S. A., Magnitogidrodinamicheskie zadachi ob ustanovivsheisia konvektsii v vertikal'nykh kanalakh (Magnetohydrodynamic problems on steady convection in vertical channels). PTMF, No. 1, 1962.
7. Whitehead, C. S., On a generalization of the functions ber x, bei x, ker x, kei x. Quart. J. Pure and Appl. Math., Vol. 42, No. 4, 1911.
8. Luke, J. L., Integrals of Bessel functions. McGraw Hill, New York, 1962.

Translated by L. K.